Probabilistic Methods in Combinatorics

Solutions to Assignment 13

Problem 1. Let $\mathcal{F}_1, \ldots, \mathcal{F}_k$ be all decreasing or all increasing families of subsets of $\{0,1\}^N$ and let $\mathbb{P}$ be a product probability space on $\{0,1\}^N$. Then,

$$\mathbb{P}[\mathcal{F}_1 \cap \mathcal{F}_2 \cdots \cap \mathcal{F}_k] \geq \prod_{i=1}^{k} \mathbb{P}[\mathcal{F}_i].$$

Solution. We prove the statement in case all $\mathcal{F}_i$ are decreasing, the other case is analogous. We proceed by induction on k. For $k = 1$ there is nothing to prove. For $k = 2$, this is the FKG inequality from the lecture notes. Now, suppose $k > 2$ and the statement holds for $k - 1$. First, we show that $\mathcal{F}' = \mathcal{F}_1 \cap \cdots \cap \mathcal{F}_{k-1}$ is a decreasing family. Indeed, suppose $A \in \mathcal{F}'$ and $B \subseteq A$. Then, by definition $A \in \mathcal{F}_i$, for all $i \in [k-1]$. Since $\mathcal{F}_i$ is decreasing, we have $B \in \mathcal{F}_i$ and it follows that $B \in \mathcal{F}'$, as needed. Hence, we can apply the FKG inequality for the two families $\mathcal{F}'$ and $\mathcal{F}_k$ and the induction hypothesis to conclude

$$\Pr[\mathcal{F}_1 \cap \cdots \cap \mathcal{F}_k] \geq \Pr[\mathcal{F}' \cap \mathcal{F}_k] \geq \Pr[\mathcal{F}'] \Pr[\mathcal{F}_k] \geq \prod_{i=1}^{k} \Pr[\mathcal{F}_i],$$

finishing the proof.

Problem 2. Let G be a graph with m edges, and let S be a random set of vertices of G obtained by picking each vertex independently with probability $1/2$. Prove that the probability that S is an independent set in G is at least $(3/4)^m$.

Solution. For every edge uv let A_{uv} be the event that at most one of u and v is in S. Note that this event is decreasing: if A_{uv} holds for S and some vertex is removed from S, then the event still holds. Denote the edges of G by $e_1, \ldots, e_m$. Then the event $A_{e_1} \cap \ldots \cap A_{e_i}$ is decreasing for every i. Thus, by the FKG inequality in the form proved in Problem 1 we find that

$$\mathbb{P}[S \text{ is independent}] = \mathbb{P}[A_{e_1} \cap \ldots \cap A_{e_m}] \geq \mathbb{P}[A_{e_1}] \cdot \ldots \cdot \mathbb{P}[A_{e_m}] = \left(\frac{3}{4}\right)^m,$$

where we used the fact that $\mathbb{P}[A_{e_i}] = 1 - (1/2)^2 = 3/4$ (because A_{e_i} holds unless both its vertices are in S).

Problem 3. A family of subsets $\mathcal{F}$ is called *intersecting* if $A \cap B \neq \emptyset$ for every $A, B \in \mathcal{F}$. Let $\mathcal{F}_1, \ldots, \mathcal{F}_k$ be intersecting families of subsets of $[n] := \{1, \ldots, n\}$. Show that $|\mathcal{F}_1 \cup \ldots \cup \mathcal{F}_k| \leq 2^n - 2^{n-k}$.

Solution. Replace each $\mathcal{F}_i$ with its up-closure $\mathcal{H}_i$ in $2^{[n]}$, i.e.

$$\mathcal{H}_i = \{A \subseteq [n] : \exists B \in \mathcal{F}_i \text{ s.t. } B \subseteq A\}.$$

Note that since $\mathcal{F}_i$ is intersecting, so is $\mathcal{H}_i$. Moreover, in order to prove that $|\mathcal{F}_1 \cup \ldots \cup \mathcal{F}_k| \leq 2^n - 2^{n-k}$, it suffices to show that $|\mathcal{H}_1 \cup \ldots \cup \mathcal{H}_k| \leq 2^n - 2^{n-k}$, because $\mathcal{F}_i \subseteq \mathcal{H}_i$.

Now pick a set $X \subseteq [n]$ uniformly at random, i.e. each element of $[n]$ is in X with probability $1/2$, independently, and let E_i be the event that $X \notin \mathcal{H}_i$. Note that the event E_i is decreasing, as $\mathcal{H}_i$ is increasing (that is why we replaced $\mathcal{F}_i$ by its up-closure). By the version of FKG inequality proved in Problem 1,

$$\mathbb{P}[E_1 \cap \ldots \cap E_k] \geq \mathbb{P}[E_1] \cdot \ldots \cdot \mathbb{P}[E_k] \geq \left(\frac{1}{2}\right)^k,$$

where we used the fact that $\mathbb{P}[E_i] \geq 1/2$, which follows from the fact that $|\mathcal{H}_i| \leq 2^{n-1}$. Indeed, because $\mathcal{H}_i$ is intersecting, it contains at most one of $A, [n] \setminus A$ for every set $A \subseteq [n]$. Finally, by the defintion of the events E_i, we have

$$|\mathcal{H}_1 \cup \ldots \cup \mathcal{H}_k| = 2^n \left(1 - \mathbb{P}[E_1 \cap \ldots \cap E_k]\right) \leq 2^n(1 - 2^{-k}),$$

as required.

Problem 4. Show that the probability that in the random graph $G(2k, 1/2)$ the maximum degree is at most $k - 1$ is at least $1/4^k$.

Solution. Let $G \sim G(2k, 1/2)$ and let $V(G) = [2k]$. Now consider an arbitrary vertex $v \in [2k]$. Then, the degree of v is distributed as $\mathrm{Bin}(2k-1, 1/2)$. The distribution of $\mathrm{Bin}(2k-1, 1/2)$ is symmetric around the mean $\frac{2k-1}{2}$. Note that $(2k-1)/2$ is not an integer. Hence $\Pr[d(v) \leq k-1] = \Pr[\mathrm{Bin}(2k-1, 1/2) < (2k-1)/2] = \Pr[\mathrm{Bin}(2k-1, 1/2) > (2k-1)/2] = 1/2$. For every $v \in [2k]$, let $\mathcal{F}_v$ denote the family of subsets of $\{0,1\}^{\binom{[2k]}{2}}$ representing all graphs on the vertex set $[2k]$ in which v has degree at most $k-1$. It is easy to see that $\mathcal{F}_v$ is a decreasing family since by removing edges from a graph we can only decrease the degree of v. Applying the inequality from Problem 1 for sets $\mathcal{F}_1, \ldots, \mathcal{F}_{2k}$, we have

$$\Pr[\Delta(G) \leq k-1] = \Pr[G \in \mathcal{F}_1 \cap \mathcal{F}_2 \cap \cdots \cap \mathcal{F}_{2k}] \geq (1/2)^{2k},$$

as claimed.

Problem 5. Let $S_1, \dots, S_k$ be random subsets of $\{1, \dots, n\}$, where each set S_i contains an element $x \in \{1, \dots, n\}$ with probability $1/\sqrt{n}$ and all of these choices are independent. Prove that with probability at least $(1-1/e)^{\binom{k}{2}}$, we have for every $1 \le i < j \le k$, $S_i \cap S_j \neq \emptyset$.

Solution. Let $\Omega = \{0,1\}^{kn}$, where the coordinate $tn+j$ for $j \in [n-1]$ represents whether the set S_{t+1} contains the element $j+1$. It is clear that on this product probability space the events $S_i \cap S_j \neq \emptyset$ are increasing events. Thus it remains to show that for fixed i, j satisfying $1 \le 1 < j \le k$, we have $\mathbb{P}(S_i \cap S_j \neq \emptyset) \ge 1 - 1/e$, and an application of the inequality proved in Problem 1 finishes the proof. As each S_i and S_j contains each element independently, we have

$$\mathbb{P}(S_i \cap S_j = \emptyset) = \left(1 - \frac{1}{\sqrt{n}} \cdot \frac{1}{\sqrt{n}}\right)^n = \left(1 - \frac{1}{n}\right)^n \le e^{-1}.$$

Taking the complement of this event gives the desired inequality.